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On packings of squares and rectangles

Derek Jennings

Department of Mathematics, University of Southampton, Southampton, Hampshire, SO9 5NH, UK

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Abstract

This paper improves the previous bound (Jennings, in press), from $\frac{133}{132}$ to $\frac{204}{203}$, concerning the smallest square into which all of the rectangles of size $1/n \times 1/(n+1)$, $n=1,2,3,\ldots$, can be packed. It also investigates the problem of determining the rectangle of smallest area into which the squares of side $1/n$, $n=3,5,7,\ldots$, can be packed.

Leo Moser notes that $\sum_{n=1}^{\infty} 1/n(n+1)=1$ and asked if the rectangles of size $1/n \times 1/(n+1)$, $n=1,2,3,\ldots$, can be packed into the unit square. Meir and Moser showed that they can be packed into a square of side $\frac{31}{30}$ [3], and in [2] this bound was improved to $\frac{133}{132}$. In this paper it is proved that they can all be packed into a square of side $\frac{204}{203}$. Also we note that $\sum_{n=1}^{\infty} 1/(2n+1)^2=(\pi^2/8)-1$, and ask do the squares of side $1/n$, $n=3,5,7,\ldots$, fit into some rectangle of area $(\pi^2/8)-1$? In this paper it is proved that they can be packed into a rectangle of area $\frac{32}{135}$, which is less than $(\pi^2/8)-1+\frac{1}{299}$. The latter result appears to be the more difficult of the two to improve upon. We start with the following theorem.

Theorem 1. *All the rectangles of size $1/n \times 1/(n+1)$, $n=1,2,3,\ldots$, can be packed into a square of side $\frac{204}{203}$.*

Proof. Let the rectangle of size $1/n \times 1/(n+1)$ be represented by Q_n. Then Fig. 1 shows how the inequality

$$\frac{1}{3n-1}+\frac{1}{3n}+\frac{1}{3n+1}>\frac{1}{n} \tag{1}$$

is used to pack the rectangles

$$\{Q_n\},\ \{Q_{3n-1}, Q_{3n}, Q_{3n+1}\},\ \{Q_{9n-4}, Q_{9n-3}, \ldots, Q_{9n+4}\},\ \ldots,$$

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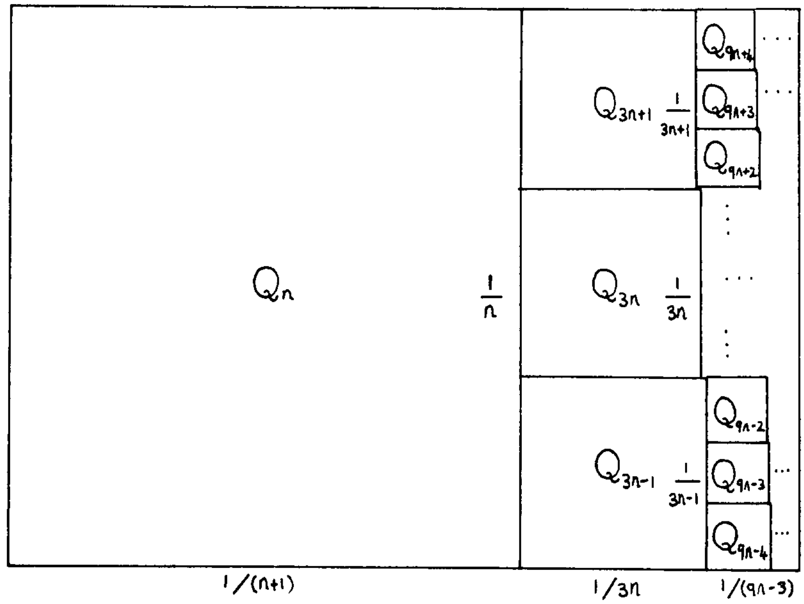

Fig. 1

into a rectangle of size (horizontal × vertical dimension)

$$\left\{\frac{1}{n+1}+\frac{1}{3n}+\frac{1}{9n-3}+\frac{1}{27n-12}+\frac{1}{81n-39}+\cdots\right\}\times\lim_{m\to\infty}\sum_{i=-(3^m-1)/2}^{(3^m-1)/2}\frac{1}{(3^m n+i)}. \quad (2)$$

Now the length (horizontal dimension) in (2) is bounded above by

$$\frac{1}{n+1}+\frac{1}{3n}+\frac{1}{9(n-1)}\left\{1+\frac{1}{3}+\frac{1}{9}\cdots\right\}=\frac{1}{n+1}+\frac{1}{3n}+\frac{1}{6(n-1)}$$

and the width (vertical dimension) is equal to

$$\int_{n-1/2}^{n+1/2}\frac{\mathrm{d}x}{x}=\log\left(\frac{2n+1}{2n-1}\right).$$

Let the rectangle of size $\{1/(n+1)+1/3n+1/6(n-1)\}\times\log\{(2n+1)/(2n-1)\}$ be represented by V_n. Then into V_n we can pack the rectangles

$$\{Q_n\},\ \{Q_{3n-1}, Q_{3n}, Q_{3n+1}\},\ \{Q_{9n-4}, Q_{9n-3}, \ldots, Q_{9n+4}\},\ \ldots,$$

and into V_{n+1}, we can pack

$$\{Q_{n+1}\},\ \{Q_{3n+2}, Q_{3n+3}, Q_{3n+4}\},\ \{Q_{9n+5}, Q_{9n+6}, \ldots, Q_{9n+13}\},\ \ldots,$$

and so on, until we reach V_{3n-2}, into which we can pack

$$\{Q_{3n-2}\},\ \{Q_{9n-7}, Q_{9n-6}, Q_{9n-5}\},\ \{Q_{27n-22}, Q_{27n-21}, \ldots, Q_{27n-14}\},\ \ldots$$

Therefore we have shown that all the Q_i, $i=1,2,3,\ldots$, can be packed into the $3n-2$ rectangles

$$\{Q_1, Q_2, \ldots, Q_{n-1}, V_n, V_{n-1}, \ldots, V_{3n-2}\}. \tag{3}$$

Fig. 2, with $n=11$, shows a packing of $\{Q_1, Q_2, \ldots, Q_{10}, V_{11}, \ldots, V_{31}\}$ into a square of side $\frac{191}{190}$. The numbers k on the edges of the diagram indicate that the width measured to this edge is less than $1+1/k$. A minus sign indicates that the edge is within a distance of $1/k$ from the edge of the unit square.

By using more terms of the infinite series in (2), determining the length of V_n, we can slightly improve upon the bound $\frac{191}{190}$. If we approximate the remainder from the sixth term onwards (instead of from the third term onwards) we find that the length of V_{18} is less than

$$\frac{1}{19}+\frac{1}{54}+\frac{1}{159}+\frac{1}{474}+\frac{1}{1419}+\frac{1}{2754}=0.08061694\ldots,$$

giving a total width in Fig. 2, at the V_{18} edge, of less than $\frac{204}{203}$. Similarly, the other critical edge at V_{30} can be reduced to less than $\frac{209}{208}$, by noting that the lengths of V_{17} and V_{30} are less than

$$\left(\frac{1}{18}+\frac{1}{51}+\frac{1}{150}+\frac{1}{288}\right) \text{ and } \left(\frac{1}{31}+\frac{1}{90}+\frac{1}{267}+\frac{1}{522}\right),$$

respectively. This completes the proof of the theorem. □

Taking more terms of the infinite series in (2) will not reduce this result to 205/204.

To analyse the efficiency of this packing algorithm we note that the area of the rectangle V_n, is less than

$$\frac{3}{2n^2}-\frac{5}{6n^3}+\frac{31}{24n^4}.$$

Hence the total area of the rectangles in (3) is less than

$$\sum_{i=1}^{n-1}\frac{1}{i(i+1)}+\sum_{i=n}^{3n-2}\left\{\frac{3}{2i}-\frac{5}{6i^3}+\frac{31}{24i^4}\right\}\sim 1+\frac{7}{54n^2}+\frac{169}{972n^3}+\mathrm{O}\left(\frac{1}{n^4}\right),$$

using the Euler–Maclaurin summation formula. This is more efficient than the packing algorithm given in [2] (where the inequality $1/2n+1/(2n+1)<1/n$ was used) by a factor of about $\frac{12}{7}$. Since in [2] all the Q_i, $i=1,2,3,\ldots$, were first packed into $2n$ rectangles by an algorithm which included an excess area of approximately $1/2n^2$, and in this paper all the Q_i are first packed into $3n-2$ rectangles by an algorithm which includes an excess area of approximately $7/54n^2$. Hence when packings of equivalent numbers of rectangles are considered the algorithm in this paper includes an excess area of only $\frac{7}{12}$ of that in [2].

A similar type of packing argument is used to prove the next theorem.

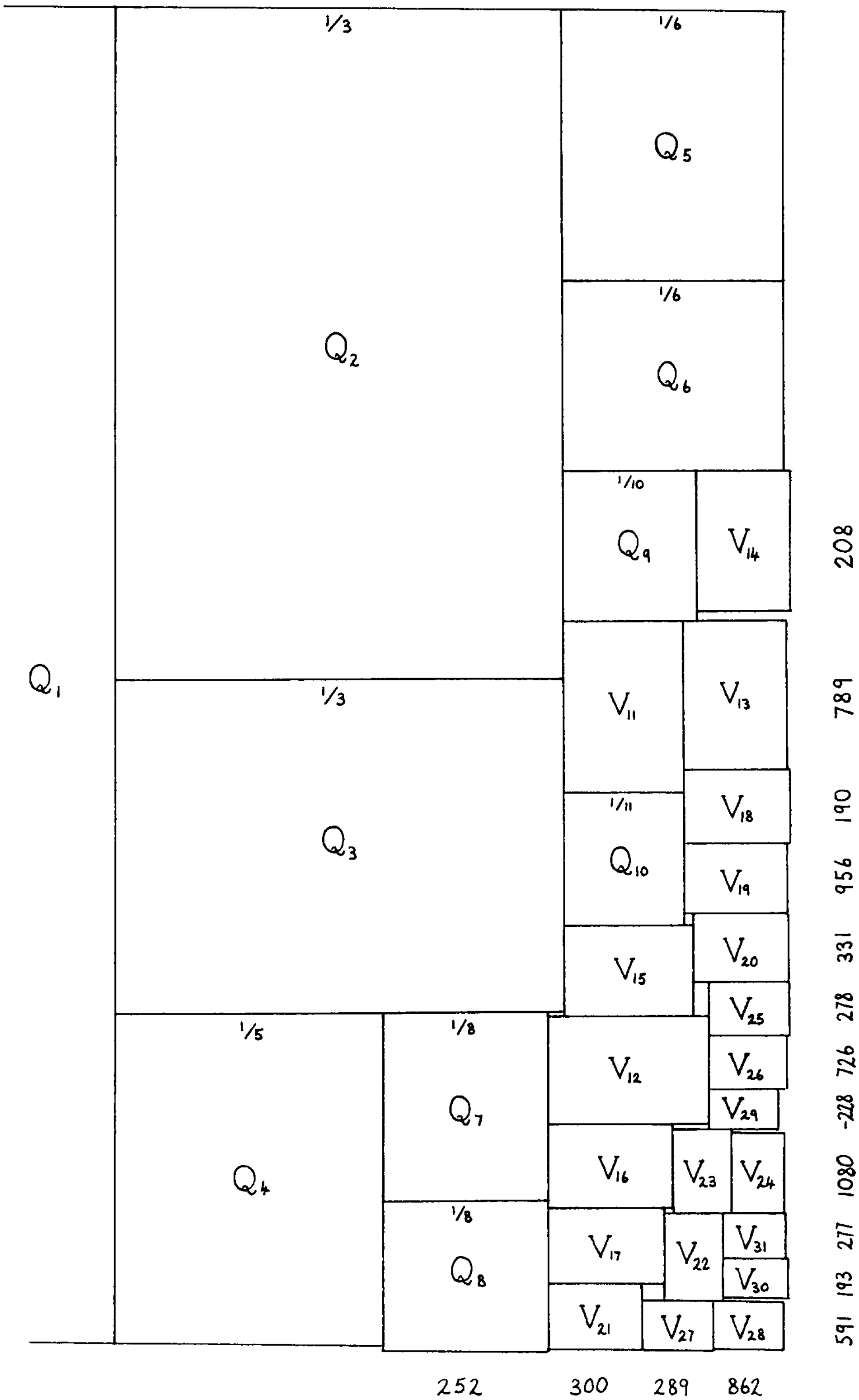
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Fig. 2

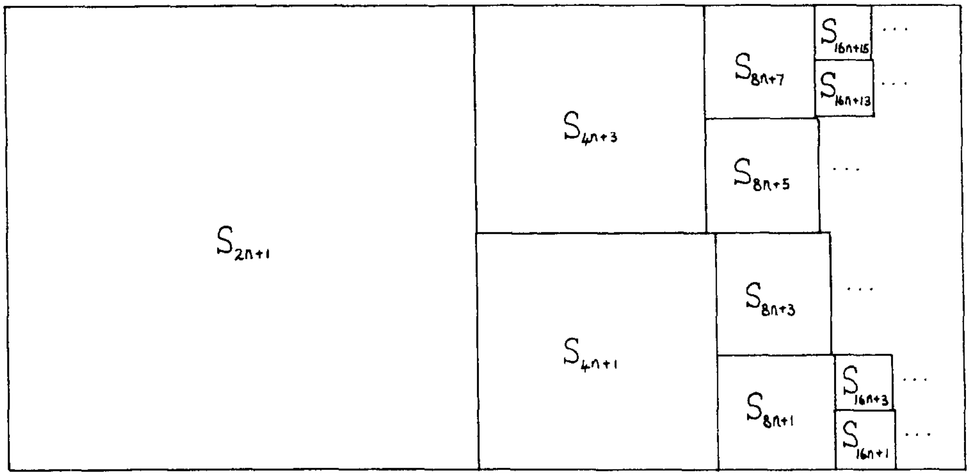

Fig. 3

Theorem 2. *All the squares of side* $1/n$, $n=3,5,7,\ldots$, *can be packed into a rectangle of area* $32/135$.

The dimensions of the enclosing rectangle are

$$\left(\frac{1}{3}+\frac{1}{5}\right)\times\left(\frac{1}{3}+\frac{1}{9}\right).$$

Note that

$$\frac{32}{135}<\left(\frac{\pi^2}{8}-1\right)+\frac{1}{299}.$$

Proof. Let the square of side $1/n$ be represented by S_n. Then Fig. 3 shows how the inequality

$$\frac{1}{2n-1}+\frac{1}{2n+1}>\frac{1}{n} \tag{4}$$

is used to pack the squares

$$\{S_{2n+1}\},\ \{S_{4n+1},S_{4n+3}\},\ \{S_{8n+1},S_{8n+3},S_{8n+5},S_{8n+7}\},\ldots,$$

into a rectangle of size (horizontal × vertical dimension)

$$\left\{\frac{1}{2n+1}+\frac{1}{4n+1}+\frac{1}{8n+1}+\frac{1}{16n+1}+\cdots\right\}\times\lim_{m\to\infty}\sum_{i=1}^{2^{m-1}}\frac{1}{2^m n+(2i-1)}. \tag{5}$$

Now the length (horizontal dimension) in (5) is bounded by

$$\frac{1}{2n+1}+\frac{1}{4n}\left\{1+\frac{1}{2}+\frac{1}{4}\cdots\right\}=\frac{1}{2n+1}+\frac{1}{2n}$$

and the width (vertical dimension) is equal to

$$\int_0^{1/(2n+1)}\frac{\mathrm{d}x}{1-x^2}=\frac{1}{2}\log\left(\frac{1+1/(2n+1)}{1-1/(2n-1)}\right)=\frac{1}{2}\log\left(1+\frac{1}{n}\right).$$

Let the rectangle of size $\{1/2n+1/(2n+1)\}\times\frac{1}{2}\log(1+1/n)$ be represented by U_{2n+1}. Then into U_{2n+1} we can pack the squares

$$\{S_{2n+1}\},\ \{S_{4n+1},S_{4n+3}\},\ \{S_{8n+1},S_{8n+3},S_{8n+5},S_{8n+7}\},\ \ldots,$$

and into U_{2n+3} we can pack

$$\{S_{2n+3}\},\ \{S_{4n+5},S_{4n+7}\},\ \{S_{8n+9},S_{8n+11},S_{8n+13},S_{8n+15}\},\ \ldots,$$

and so on, until we reach U_{4n-1}, into which we can pack

$$\{S_{4n-1}\},\ \{S_{8n-3},S_{8n-1}\},\ \{S_{16n-7},S_{16n-5},S_{16n-3},S_{16n-1}\},\ \ldots,$$

Therefore we have shown that all the S_i, $i=3,5,7,\ldots$, can be packed into the $2n-1$ squares and rectangles

$$\{S_3,S_5,S_7,\ldots,S_{2n-1},U_{2n+1},U_{2n+3},\ldots,U_{4n-1}\}. \tag{6}$$

Fig. 4, with $n=15$, completes the proof of the theorem by showing a packing of $\{S_3,S_5,\ldots,S_{29},U_{31},\ldots,U_{59}\}$ into a rectangle of area $\frac{32}{135}$. The numbers k on the edges of the diagram indicate that the edge of the adjacent rectangle is within a distance on $1/k$ from the edge of the enclosing rectangle. □

To analyse the efficiency of this packing algorithm we note that the width of the rectangle U_{2n+1}, is

$$\frac{1}{2}\log\left(1+\frac{1}{n}\right)<\frac{1}{(2n+1)}\left\{1+\frac{1}{12n(n+1)}\right\}.$$

Hence the total area of the squares and rectangles in (6) is less than

$$\sum_{i=1}^{n-1}\frac{1}{(2i+1)^2}+\sum_{i=n}^{2n-1}\frac{1}{2i+1}\left\{1+\frac{1}{12i(i+1)}\right\}\left\{\frac{1}{2i}+\frac{1}{2i+1}\right\}.$$

The right-hand side summation is equal to

$$\frac{1}{16n}+\frac{2}{3}\sum_{k=n}^{2n-1}\frac{1}{k}-\frac{2}{3}\sum_{k=2n}^{4n-1}\frac{1}{k}-\frac{1}{8}\sum_{k=n}^{2n-1}\frac{1}{k^2}+\frac{2}{3}\sum_{k=2n}^{4n-1}\frac{1}{k^2}\sim\frac{1}{4n}+\frac{3}{64n^2}-\frac{7}{1152n^3}+\mathrm{O}\left(\frac{1}{n^4}\right),$$

using Euler–Maclaurin summation, and

$$\sum_{i=1}^{n-1}\frac{1}{(2i+1)^2}\sim\left(\frac{\pi^2}{8}-1\right)-\frac{1}{4n}+\frac{1}{48n^3}+\mathrm{O}\left(\frac{1}{n^5}\right).$$

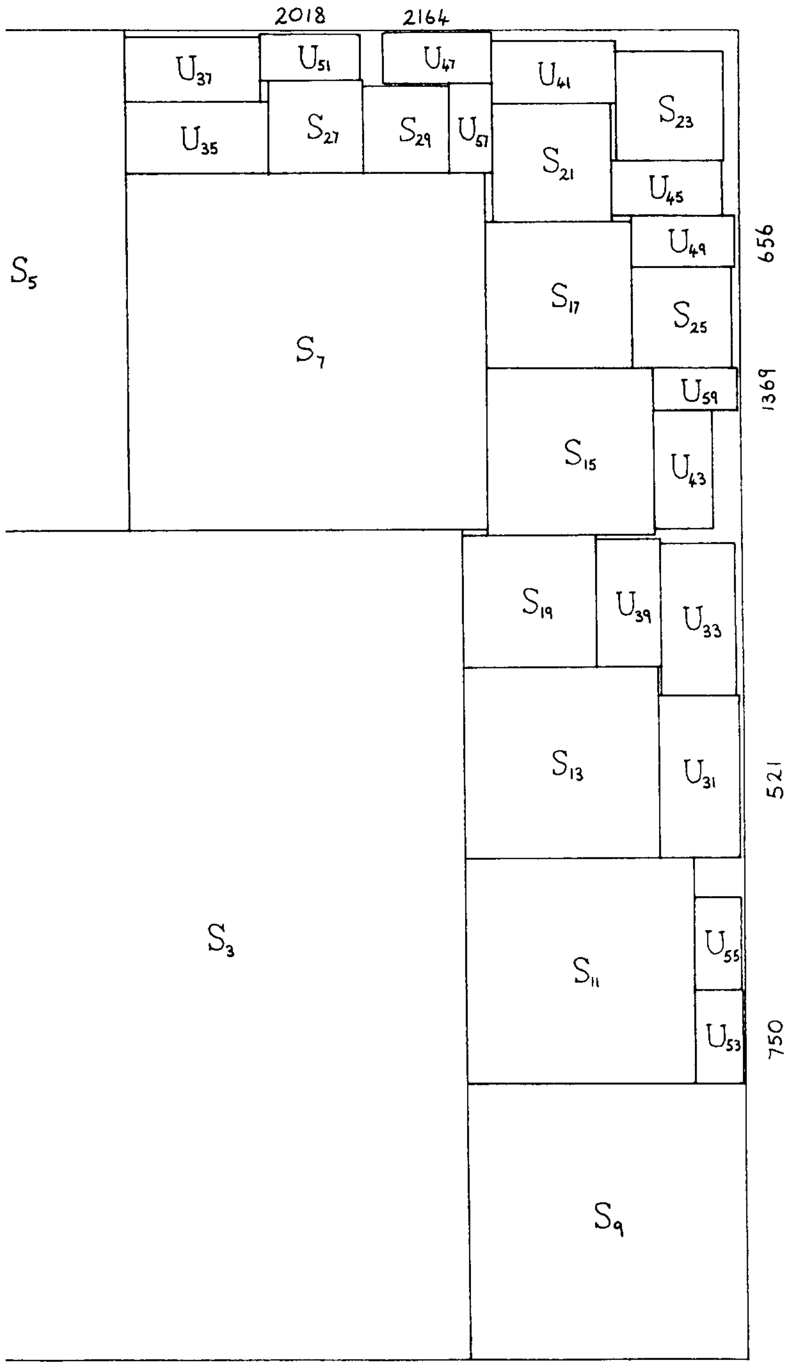
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Fig. 4

Hence the total area of the squares and rectangles in (6) is less than

$$\left(\frac{\pi^2}{8}-1\right)+\frac{3}{64n^2}+\frac{17}{1152n^3}+O\left(\frac{1}{n^4}\right).$$

Therefore the excess area included when packing $2n-1$ rectangles, by this algorithm, is about $3/64n^2$. This is over six times as efficient as the algorithm used in Theorem 1.

Improvements on Theorem 2, by the methods of this paper, may be difficult, since the squares S_3, S_5 and S_7 would have to be aligned. This would give one dimension of $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$, requiring the other dimension to be less than $(\frac{1}{3}+\frac{1}{81})$. Hence a value of n much larger than the $n=15$ used to derive Theorem 2, would be needed.

Of course, the problem of finding the smallest $\varepsilon \geqslant 0$, such that all the rectangles of size $1/n \times 1/(n+1)$, $n=1,2,3,\ldots$, can be packed into a square of side $1+\varepsilon$, is still unsolved [1]. Whether $\varepsilon>0$ or $\varepsilon=0$ is still an open question. Similarly, the problem of finding the smallest $\varepsilon \geqslant 0$, such that all the squares of side $1/n$, $n=3,5,7,\ldots$, can be packed into a rectangle of area $(\pi^2/8)-1+\varepsilon$ has not been solved. Could the packing given in this paper actually be optimum? It may be hard to prove that it is not.

References

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